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Exercices de Préparation au DS N°3

Systèmes, trigonométrie, Complexes

Ex1

pivot de Gauss : on l'utilise pour éliminer x dans L_2 et L_3

a) $\begin{cases} x+2y+z=3 & (L_1) \\ -x+y+z=0 & (L_2) \\ 2x+3y+4z=13 & (L_3) \end{cases} \iff \begin{cases} x+2y+z=3 & (L_1) \\ 3y+2z=3 & (L_2)+(L_1) \\ -y+2z=7 & (L_3)-2(L_1) \end{cases}$

pivot de Gauss : pour éliminer y dans (L_3)

$\iff \begin{cases} x+2y+z=3 & (L_1) \\ 3y+2z=3 & (L_2) \\ 8z=24 & 3(L_3)+(L_2) \end{cases}$

on a obtenu un système triangulaire.

$\Rightarrow$ on peut déterminer z, puis y puis x

$\Rightarrow z=\frac{24}{8}=3$

$3y=3-2z=3-6=-3 \Rightarrow y=-1$

$x=3-2y-z=3+2-3=2$

d'où $S=\{(2,-1,3)\}$

Chaque équation (L_1, L_2 et L_3) représente un plan dans l'espace.
Les 3 plans se coupent en un point $A(2,-1,3)$

b) $\begin{cases} x+2y+z=3 & (L_1) \\ -x+y+z=0 & (L_2) \\ 2x+y=3 & (L_3) \end{cases} \iff \begin{cases} x+2y+z=3 & (L_1) \\ 3y+2z=3 & (L_2)+(L_1) \\ -3y-2z=-3 & (L_3)-2(L_1) \end{cases}$

$\iff \begin{cases} x+2y+z=3 & (L_1) \\ 3y+2z=3 & (L_2) \\ 0=0 & (L_3)+(L_2) \end{cases} \iff \begin{cases} x+2y+z=3 & (L_1) \\ 3y+2z=3 & (L_2) \end{cases}$

2 équations
3 inconnues

on met z en paramètre $\begin{cases} x+2y=3-z \\ 3y=3-2z \end{cases}$

d'où $y=1-\frac{2}{3}z$

$x=3-z-2y=3-z-2+\frac{4}{3}z$

$x=1+\frac{1}{3}z$

Equation paramétrique d'une droite dans l'espace

Finalement $\begin{cases} x = 1 + \frac{1}{3} z \\ y = 1 - \frac{2}{3} z \\ z = 0 + 1 z \end{cases}$

Les 3 plans se coupent selon la droite $D(A, \vec{u})$ avec $A(1,1,0)$ et $\vec{u}\left(\frac{1}{3}, -\frac{2}{3}, 1\right)$

$$S = \left\{ \left(1 + \frac{1}{3} z, 1 - \frac{2}{3} z, z\right), z \in \mathbb{R} \right\}$$

c) $\begin{cases} x + 2y + z = 3 & (L_1) \\ -x + y + z = 0 & (L_2) \\ 2x + y = 5 & (L_3) \end{cases} \iff \begin{cases} x + 2y + z = 3 & (L_1) \\ 3y + 2z = 3 & (L_2) + (L_1) \\ -3y - 2z = -1 & (L_3) - 2(L_1) \end{cases}$

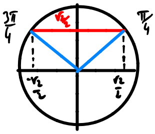
$\iff \begin{cases} x + 2y + z = 3 & (L_1) \\ 3y + 2z = 3 & (L_2) \\ 0 = 2 & (L_3) + (L_2) \end{cases}$ impossible ! $\quad S = \emptyset$

Les 3 plans n'ont aucun point d'intersection.

$\sin x = \sin y \iff \begin{cases} x = y \, (2\pi) \\ \text{ou} \\ x = \pi - y \, (2\pi) \end{cases}$

Ex 2 : 1) $\sin x = \frac{\sqrt{2}}{2} \iff \sin x = \sin \frac{\pi}{4} \iff x = \frac{\pi}{4} \, (2\pi)$ ou $x = \pi - \frac{\pi}{4} \, (2\pi)$

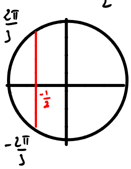
$x = \frac{3\pi}{4} \, (2\pi)$



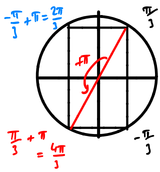
$$S_{\mathbb{R}} = \left\{ \frac{\pi}{4} \, [2\pi] \, ; \, \frac{3\pi}{4} \, [2\pi] \right\}$$

$\cos x = \cos y \iff x = \pm y \, (2\pi)$

2) $\cos(2x) = -\frac{1}{2} \iff \cos(2x) = \cos\left(\frac{2\pi}{3}\right) \iff 2x = \pm \frac{2\pi}{3} \, (2\pi) \iff x = \pm \frac{\pi}{3} \, (\pi)$ ⚠ modulo divisé par 2



$$S_{\mathbb{R}} = \left\{ \pm \frac{\pi}{3} \, (\pi) \right\}$$



Représentation des solutions sur le cercle trigonométrique

$\frac{\pi}{3} + \pi = \frac{4\pi}{3} \notin]-\pi, \pi] \Rightarrow \frac{4\pi}{3} - 2\pi = -\frac{2\pi}{3}$

$-\frac{\pi}{3} + \pi = \frac{2\pi}{3} \in]-\pi, \pi]$

$$S_{]-\pi,\pi]} = \left\{ \frac{\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}, -\frac{2\pi}{3} \right\}$$

3) $\cos x + \sqrt{3} \sin x = 1$

$A = 1, B = \sqrt{3} \Rightarrow C = \sqrt{1+3} = 2$

$A \cos x + B \sin x = C \cos(x - \varphi)$

avec $C = \sqrt{A^2 + B^2}$ $\quad \cos\varphi = \frac{A}{C}$ $\quad \sin\varphi = \frac{A}{C}$

$$\cos x + \sqrt{3}\sin x = 2\left(\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x\right)$$
$$= 2\left(\cos\frac{\pi}{3}\cos x + \sin\frac{\pi}{3}\sin x\right) \qquad \cos(a-b) = \cos a\cos b + \sin a \sin b$$
$$= 2\cos\left(x - \frac{\pi}{3}\right)$$

donc $\cos x + \sqrt{3}\sin x = 1 \iff 2\cos\left(x-\frac{\pi}{3}\right) = 1 \iff \cos\left(x - \frac{\pi}{3}\right) = \frac{1}{2}$

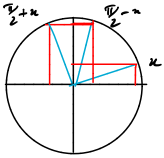
$$\iff \cos\left(x - \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) \qquad \cos x = \cos y \iff x = \pm y \ [2\pi]$$

$$\iff x - \frac{\pi}{3} = \frac{\pi}{3} \ [2\pi] \text{ ou } x - \frac{\pi}{3} = -\frac{\pi}{3} \ [2\pi]$$

$$\iff x = \frac{2\pi}{3} \ [2\pi] \text{ ou } x = 0 \ [2\pi]$$

$$S_{\mathbb{R}} = \left\{ \frac{2\pi}{3} [2\pi], 0 [2\pi] \right\} \text{ et } S_{[0,2\pi]} = \left\{ \frac{2\pi}{3}, 0, 2\pi \right\}$$

4) $\sin\left(x + \frac{\pi}{4}\right) = -\cos x$



$$\begin{cases} \cos\left(\frac{\pi}{2} - x\right) = \sin x \\ \sin\left(\frac{\pi}{2} - x\right) = \cos x \end{cases}$$

$$\begin{cases} \cos\left(\frac{\pi}{2} + x\right) = -\sin x \\ \sin\left(\frac{\pi}{2} + x\right) = \cos x \end{cases}$$

$$\iff \cos x = -\sin\left(x + \frac{\pi}{4}\right)$$

$$\iff \cos x = \cos\left(\frac{\pi}{2} + x + \frac{\pi}{4}\right)$$

$$\iff x = \frac{\pi}{2} + x + \frac{\pi}{4} \ [2\pi] \text{ ou } x = -\frac{\pi}{2} - x - \frac{\pi}{4} \ [2\pi]$$

$$\iff 0 = \frac{3\pi}{4} \ [2\pi] \qquad \text{ou} \qquad 2x = -\frac{3\pi}{4} \ [2\pi]$$

impossible $\qquad \iff x = -\frac{3\pi}{8} \ [\pi]$

$$S_{\mathbb{R}} = \left\{ -\frac{3\pi}{8} [\pi] \right\}$$

Ex 3

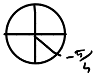
forme algébrique

$$a = \frac{1}{1+j} \times \frac{1-j}{1-j} = \frac{1-j}{1^2+1^2} = \frac{1}{2} - \frac{1}{2}j$$

$z\bar{z} = (x+jy)(x-jy) = x^2+y^2$

$$|a| = \sqrt{\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$\left.\begin{aligned} \cos\theta &= \frac{x}{r} = \frac{\frac{1}{2}}{\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{2} \\ \sin\theta &= \frac{y}{r} = \frac{-\frac{1}{2}}{\frac{1}{\sqrt{2}}} = -\frac{\sqrt{2}}{2} \end{aligned}\right\} \theta = -\frac{\pi}{4}\ (2\pi)$$



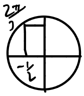
forme exponentielle

2e méthode : $a = \frac{1}{2} - \frac{1}{2}j = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}j\right) = \frac{1}{\sqrt{2}} e^{-j\frac{\pi}{4}}$

$b = -e^{j\frac{\pi}{3}}$ ⚠ ce n'est pas la forme exponentielle à cause du signe ⊖

$b = e^{j\pi} e^{-j\frac{\pi}{3}}$ on a remplacé le ⊖ par $e^{j\pi} = -1$

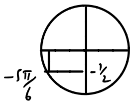
$$b = \underbrace{e^{2j\frac{\pi}{3}}}_{\text{forme exp.}} = \cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3} = \underbrace{-\frac{1}{2} + j\frac{\sqrt{3}}{2}}_{\text{forme algébrique}}$$



$$c = 2j(-1+\sqrt{3}j) = -2j + 2\sqrt{3}j^2 = -2j - 2\sqrt{3} = \underbrace{-2\sqrt{3} - 2j}_{\text{forme algébrique}}$$

$$|c| = \sqrt{4\times3+4} = 4$$

$$c = 4\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}j\right) = \underbrace{4e^{-j\frac{5\pi}{6}}}_{\text{forme exp.}}$$



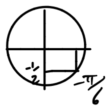
$$d = \left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right)^5 = \left(e^{j\frac{\pi}{4}}\right)^5 = e^{5j\frac{\pi}{4}} = \underbrace{e^{-3j\frac{\pi}{4}}}_{\text{forme exp}} \quad \text{car } \frac{5\pi}{4} - 2\pi = -\frac{3\pi}{4}$$

$$d = \cos\left(-\frac{3\pi}{4}\right) + j\sin\left(-\frac{3\pi}{4}\right) = \underbrace{-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}j}_{\text{forme algébrique}}$$

Ex 4

$|\sqrt{3}-j|=2$

1) a) $z^2 = \sqrt{3} - j = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}j\right) = 2e^{-j\frac{\pi}{6}}$



$\iff z = \left(\sqrt{2}e^{-j\frac{\pi}{12}}\right)^2 \iff z = \pm\sqrt{2}e^{-j\frac{\pi}{12}}$

b) $z^2 - (4j+1)z - 3 + 3j = 0$

| $a = 1$
| $b = -(4j+1)$
| $c = -3+3j$

$\Delta = b^2 - 4ac = (4j+1)^2 - 4(-3+3j)$

$\Delta = -16 + 8j + 1 + 12 - 12j = -3 - 4j$

Méthode 1 : on résoud $z^2 = -3-4j \iff (x+iy)^2 = -3-4j$

$\iff x^2 + 2ixy - y^2 = -3 - 4j$

$\iff \begin{cases} (L_1) & x^2 - y^2 = -3 \\ (L_2) & 2xy = -4 \\ (L_3) & x^2 + y^2 = 5 \end{cases}$

partie réelle

partie imaginaire

module : En effet $|z|^2 = x^2 + y^2 = |-3-4j| = 5$

$(L_1) + (L_3)$ $2x^2 = 2 \iff x^2 = 1 \iff x = \pm 1$

$(L_3) - (L_1)$ $2y^2 = 8 \iff y^2 = 4 \iff y = \pm 2$

De plus, $xy = -2 < 0$
x et y sont de signe contraire

donc $z = \pm(1-2j)$

on choisit par exemple $\delta = 1-2j$

Méthode 2 : On écrit -3 comme la différence de 2 carrés : $-3 = 1 - 4$

$\Delta = -3 - 4j = 1 - 4j - 4 = 1 - 4j + (2j)^2 = (1-2j)^2$ (identité remarquable)

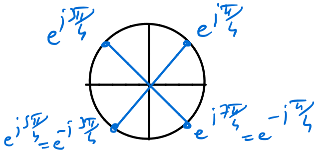
donc $\delta = 1 - 2j$

d'où $\begin{cases} z_1 = \dfrac{-b+\delta}{2a} = \dfrac{4j+1+1-2j}{2} = \dfrac{2+2j}{2} = 1+j \\ \\ z_2 = \dfrac{-b-\delta}{2a} = \dfrac{4j+1-1+2j}{2} = 3j \end{cases}$

2) $z^4 = -1$ on pose $z = re^{j\theta}$

$\Leftrightarrow r^4 e^{4j\theta} = e^{j\pi}$ on identifie module et argument

$\Leftrightarrow \begin{cases} r^4 = 1 \\ 4\theta = \pi \ [2\pi] \end{cases} \Leftrightarrow \begin{cases} r = 1 \text{ car } r \geq 0 \\ \theta = \frac{\pi}{4} \ [\frac{2\pi}{4}] = \frac{\pi}{4} + \frac{2k\pi}{4} \end{cases}$



$S = \left\{ e^{j\frac{\pi}{4}} ; e^{j\frac{3\pi}{4}} ; e^{j\frac{5\pi}{4}} ; e^{j\frac{7\pi}{4}} \right\}$

<u>2e méthode</u> : $z^4 = -1 = e^{j\pi} = \left(e^{j\frac{\pi}{4}}\right)^4$ donc $z_0 = e^{j\frac{\pi}{4}}$ est solution évidente, les autres se trouvent par rotation de $\frac{2\pi}{4}$. on trouve les mêmes solutions.

1er terme — nombre de termes

3) $S = 1 \underbrace{+ e^{j\frac{\pi}{3}}}_{\times e^{j\frac{\pi}{3}}} \underbrace{+ e^{2j\frac{\pi}{3}}}_{\times e^{j\frac{\pi}{3}}} + e^{3j\frac{\pi}{3}} + \dots + e^{9j\frac{\pi}{3}} = 1 \times \dfrac{1 - q^{10}}{1 - q}$ avec $q = e^{j\frac{\pi}{3}}$

somme des termes d'une suite géométrique

$= \dfrac{1 - e^{j\frac{10\pi}{3}}}{1 - e^{j\frac{\pi}{3}}}$

or $\dfrac{10\pi}{3} - 2\pi = \dfrac{4\pi}{3}$ et $e^{j\frac{4\pi}{3}} = e^{-j\frac{2\pi}{3}}$

donc $S = \dfrac{1 - e^{-j\frac{2\pi}{3}}}{1 - e^{j\frac{\pi}{3}}} = \dfrac{1 + \frac{1}{2} + \frac{\sqrt{3}}{2}j}{1 - \frac{1}{2} - \frac{\sqrt{3}}{2}j} = \dfrac{\frac{3}{2} + \frac{\sqrt{3}}{2}j}{\frac{1}{2} - \frac{\sqrt{3}}{2}j} = \dfrac{3 + \sqrt{3}j}{1 - \sqrt{3}j} \times \dfrac{1 + \sqrt{3}j}{1 + \sqrt{3}j}$

$S = \dfrac{3 + 3\sqrt{3}j + \sqrt{3}j - 3}{1^2 + \sqrt{3}^2} = \dfrac{4\sqrt{3}j}{4} = \sqrt{3}j$

2e méthode : $S = 1 + e^{j\frac{\pi}{3}} + e^{2j\frac{\pi}{3}} + \underset{\overset{\parallel}{e^{j\pi}}}{e^{3j\frac{\pi}{3}}} + \underset{\overset{\parallel}{e^{-2j\frac{\pi}{3}}}}{e^{4j\frac{\pi}{3}}} + \underset{\overset{\parallel}{e^{-j\frac{\pi}{3}}}}{e^{5j\frac{\pi}{3}}} + \underset{\overset{\parallel}{e^{j2\pi}}}{e^{6j\frac{\pi}{3}}} + \underset{\overset{\parallel}{e^{j\frac{\pi}{3}}}}{e^{7j\frac{\pi}{3}}} + \underset{\overset{\parallel}{e^{j\frac{2\pi}{3}}}}{e^{8j\frac{\pi}{3}}} + \underset{\overset{\parallel}{e^{j\pi}}}{e^{9j\frac{\pi}{3}}}$

$S = 1 + e^{j\frac{\pi}{3}} + e^{2j\frac{\pi}{3}} + \underset{\overset{\parallel}{-1}}{e^{j\pi}} + e^{-2j\frac{\pi}{3}} + e^{-j\frac{\pi}{3}} + \underset{\overset{\parallel}{1}}{e^{j2\pi}} + e^{j\frac{\pi}{3}} + e^{j\frac{2\pi}{3}} + \underset{\overset{\parallel}{-1}}{e^{j\pi}}$

$S = 2\cos\frac{\pi}{3} + 2\cos\frac{2\pi}{3} + e^{j\frac{\pi}{3}} + e^{j\frac{2\pi}{3}}$ $\qquad e^{j\theta} + e^{-j\theta} = 2\cos\theta$

$S = 2 \times \frac{1}{2} + 2 \times -\frac{1}{2} + \frac{1}{2} + j\frac{\sqrt{3}}{2} - \frac{1}{2} + j\frac{\sqrt{3}}{2} = j\sqrt{3}$

4) $z = 1 - e^{2j\theta}$

<u>méthode 1</u> : on factorise par l'angle moitié

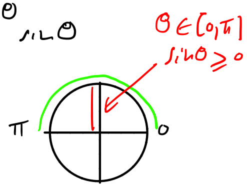
$$z = e^{j\theta}\left(e^{-j\theta} - e^{j\theta}\right) = e^{j\theta} \times -2j\sin\theta = -2j\, e^{j\theta}\sin\theta$$

<u>méthode 2</u> : on utilise la formule d'Euler : $\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

$$-2j\, e^{j\theta}\sin\theta = -2j\, e^{j\theta}\, \frac{e^{j\theta} - e^{-j\theta}}{2j} = -e^{j\theta}\left(e^{j\theta} - e^{-j\theta}\right) = -e^{j2\theta} + 1$$

on a donc $z = -2j\, e^{j\theta}\sin\theta = 2e^{-j\frac{\pi}{2}}\, e^{j\theta}\sin\theta$

$$z = \underbrace{2\sin\theta}_{|z|}\, e^{j\overbrace{\left(\theta - \frac{\pi}{2}\right)}^{\arg(z)}}$$



$|z| = 2\sin\theta \geq 0$ car $\theta \in [0,\pi]$ et $\arg(z) = \theta - \frac{\pi}{2}\ [2\pi]$