INTEGRATION

MATHS

S2

INTEGRATION PAR PARTIES

Pourquoi faire une IPP ?

$$\int \cos x \, dx = \sin x + C$$

$$\frac{1}{2}\int 2x e^{x^2} dx = \frac{1}{2} e^{x^2}$$

$u' e^{u}$ $\quad u(x) = x^2$, $u'(x) = 2x$ $\quad \int u' e^{u} = e^{u}$

Comment faire une IPP ?

$$\int_a^b u'v = [uv]_a^b - \int_a^b uv'$$

$$\int u'v = uv - \int uv'$$

pas de borne = une primitive

$$\int (x-2)\sin x \, dx = -\cos x\,(x-2) + \int \cos x \, dx$$

$$= -\cos x\,(x-2) + \sin x + C$$

Ex1 :

Calculer $I = \int_0^{\frac{\pi}{2}} \underbrace{(x-2)}_{v} \underbrace{\sin x}_{u'} \, dx$

$$\begin{cases} u'(x) = \sin x \\ v(x) = x - 2 \end{cases} \qquad \begin{cases} u(x) = -\cos x \\ v'(x) = 1 \end{cases} \qquad \int_a^b u'v = [uv]_a^b - \int_a^b uv'$$

$$I = \left[\underbrace{-\cos x}_{u}\,\underbrace{(x-2)}_{v}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \underbrace{(-\cos x)}_{u} \times \underbrace{1}_{v'} \, dx \qquad \begin{cases} \cos\frac{\pi}{2} = 0 \\ \cos 0 = 1 \end{cases}$$

$$I = 0 + 1(0-2) + \int_0^{\frac{\pi}{2}} \cos x \, dx \qquad \begin{cases} \sin\frac{\pi}{2} = 1 \\ \sin 0 = 0 \end{cases}$$

$$I = -2 + [\sin x]_0^{\frac{\pi}{2}} = -2 + \sin\frac{\pi}{2} - \sin 0$$

$$I = -2 + 1 = -1$$

Point Méthode :

Choix de u' : facile à intégrer

Choix de v : facile à dériver

On veut obtenir une intégrale + simple à calculer

ex 2 : $J = \int_1^2 \underbrace{x}_{u'} \underbrace{\ln x}_{v} \, dx$

⚠ par le choix car on ne sait pas intégrer $\ln x$

ex 3 : $K(x) = \int x^2 e^x \, dx$

⚠ 2 IPP successives

INTEGRATION PAR PARTIES

SOLUTION :

ex 2 :

$$J = \int_1^2 \underbrace{x}_{u'} \underbrace{\ln x}_{v} \, dx \qquad \begin{cases} u'(x) = x \\ v(x) = \ln x \end{cases} \qquad \begin{cases} u(x) = \frac{x^2}{2} \\ v'(x) = \frac{1}{x} \end{cases}$$

$$J = \left[\frac{x^2}{2} \ln x \right]_1^2 - \int_1^2 \frac{x^2}{2} \times \frac{1}{x} \, dx \qquad \int_a^b u'v = [uv]_a^b - \int_a^b uv'$$

$$J = 2\ln 2 - \frac{1}{2}\int_1^2 x \, dx$$

$$J = 2\ln 2 - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^2 = 2\ln 2 - \frac{1}{4}(2^2 - 1^2)$$

$$J = 2\ln 2 - \frac{3}{4}$$

ex 3 :

1ère IPP

$$K(x) = \int \underbrace{x^2}_{v} \underbrace{e^x}_{u'} \, dx \qquad \begin{cases} u'(x) = e^x \\ v(x) = x^2 \end{cases} \qquad \begin{cases} u(x) = e^x \\ v'(x) = 2x \end{cases}$$

$$K(x) = x^2 e^x - \int \underbrace{2x}_{v} \underbrace{e^x}_{u'} \, dx$$

2e IPP $\qquad \begin{cases} u'(x) = e^x \\ v(x) = 2x \end{cases} \qquad \begin{cases} u(x) = e^x \\ v'(x) = 2 \end{cases}$

$$K(x) = x^2 e^x - \left(2x e^x - \int 2e^x \, dx\right) \qquad \text{or } \int e^x \, dx = e^x + C$$

$$= x^2 e^x - 2x e^x + 2e^x + C$$

$$= e^x (x^2 - 2x + 2) + C$$

INTEGRATION

CHANGEMENT DE VARIABLE

Pourquoi faire un changement de variable ?

$$\int \cos x (\sin x)^3 \, dx = \frac{(\sin x)^4}{4} + C$$

$$\int u' u^3 = \frac{u^4}{4} + C$$

$$\int x e^x \, dx \quad \text{IPP}$$

Comment faire un changement de variable?

Exemple 1 :

Calculer $\displaystyle\int_0^1 \frac{1}{e^x+1} \, dx$

On pose $t = e^x$

Si $x = 0$ $\quad t = e^0 = 1$

Si $x = 1$ $\quad t = e^1 = e$

① Bornes

x	0	1
t	1	e

$t = f(x) \Rightarrow dt = f'(x) \, dx$

② Elément différentiel : $t = e^x \iff x = \ln t$

$dt = (e^x)' \, dx = e^x \, dx$ $\qquad dx = (\ln t)' \, dt = \frac{1}{t} \, dt$

③ On remplace tout !

1ère méthode :

$$\int_0^1 \frac{1}{e^x+1} \, dx = \int_1^e \frac{1}{t+1} \, \frac{1}{t} \, dt = \int_1^e \frac{1}{t(t+1)} \, dt$$

$$\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1} \Rightarrow \frac{1}{t(t+1)} = \frac{t+1-t}{t(t+1)} = \frac{1}{t} - \frac{1}{t+1}$$

$$\int_1^e \left(\frac{1}{t} - \frac{1}{t+1}\right) dt = \Big[\ln t - \ln(t+1)\Big]_1^e = \ln e - \ln(e+1) - \underbrace{\ln 1}_{0} + \ln 2$$

$$= \ln\left(\frac{2e}{e+1}\right)$$

2e méthode

$$\int_0^1 \frac{1}{e^x+1} \times \frac{e^x}{e^x} \, dx = \int_0^1 \frac{e^x \, dx}{e^x(e^x+1)} = \int_1^e \frac{dt}{t(t+1)}$$

la suite est identique.

Exemple 2.

Calculer $F(x) = \int \frac{\sqrt{x} + 2}{\sqrt{x}\,(x+1)}\, dx$ $\quad x > 0$

On pose $t = \sqrt{x} \iff x = t^2$

① Bornes: il n'y en a pas! Il s'agit d'un calcul de primitive.

② Elément différentiel : $t = \sqrt{x} \Rightarrow dt = \frac{1}{2\sqrt{x}} dx$

$x = t^2 \Rightarrow dx = 2t\, dt$

③ On remplace tout !

1ère méthode :

$$F(x) = \int \frac{\sqrt{x} + 2}{\sqrt{x}\,(x+1)}\, dx = \int \frac{t+2}{t\,(t^2+1)}\, 2t\, dt$$

$$F(x) = \int \frac{2(t+2)}{t^2+1}\, dt = \int \left(\frac{2t}{t^2+1} + \frac{4}{t^2+1} \right) dt$$

$$F(x) = \ln(t^2+1) + 4 \arctan t + C$$

Revenir à x dans le cas d'une primitive

$$F(x) = \ln(x+1) + 4 \arctan(\sqrt{x}) + C$$

2ème méthode : $F(x) = \int \frac{\sqrt{x} + 2}{\sqrt{x}\,(x+1)}\, dx = \int \frac{t+2}{(t^2+1)}\, 2\, dt$

On obtient le même résultat.

REGLES DE BIOCHE

Préambule :

$$u = f(x) \Rightarrow du = f'(x)\,dx$$

$d(x^2) = 2x\,dx$ car $(x^2)' = 2x$

$d(-x) = -dx$ car $(-x)' = -1$

$d(\pi - x) = -dx$ car $(\pi - x)' = -1$

$d(\pi + x) = dx$ car $(\pi + x)' = 1$

Règles de Bioche :

$$\frac{\cos x}{1 + \tan x}$$

Soit $I = \int_a^b f(x)\,dx$

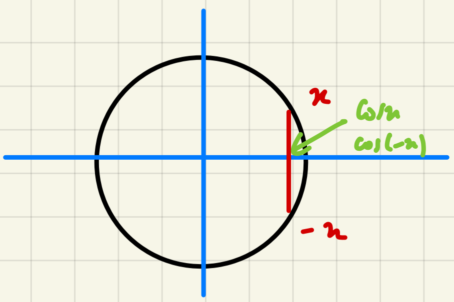
où $f(x) = F(\cos x, \sin x)$ est une fraction rationnelle en $\cos x$ et $\sin x$

Si $f(x)dx$ est invariante :

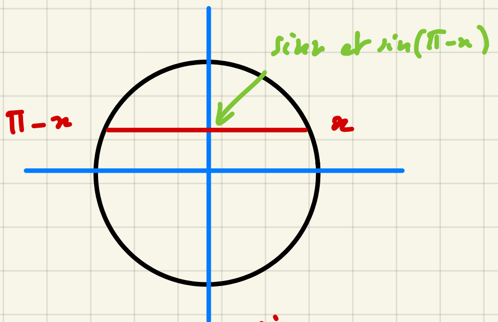
① par le changement $x \mapsto (-x)$, on pose $t = \cos x$

② par le changement $x \mapsto \pi - x$, on pose $t = \sin x$

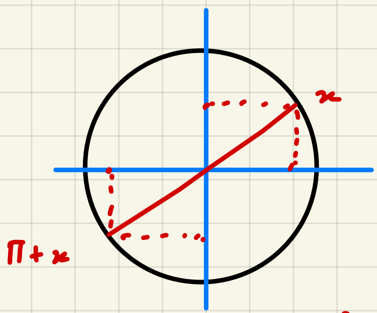
③ par le changement $x \mapsto \pi + x$, on pose $t = \tan x$



$\cos(-x) = \cos x$



$\sin(\pi - x) = \sin x$



$$\tan(\pi + x) = \frac{\sin(\pi + x)}{\cos(\pi + x)} = \frac{-\sin x}{-\cos x} = \tan x$$

Ex 1 :

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x}\,dx$$

$\int \frac{u'}{u}$

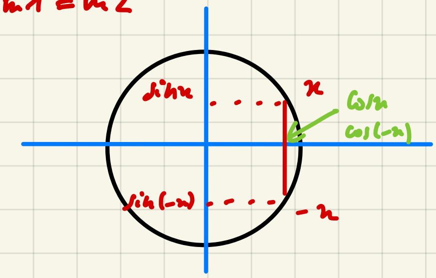
$u(x) = 1 + \sin x$

$u'(x) = \cos x$

$$I = \Big[\ln(1+\sin x)\Big]_0^{\frac{\pi}{2}} = \ln 2 - \ln 1 = \ln 2$$

$x \leftrightarrow -x$ $\qquad d(-x) = -dx$

$$\frac{\cos(-x)}{1+\sin(-x)}\,d(-x) = \frac{\cos x}{1-\sin x} \times -dx \quad ✗$$



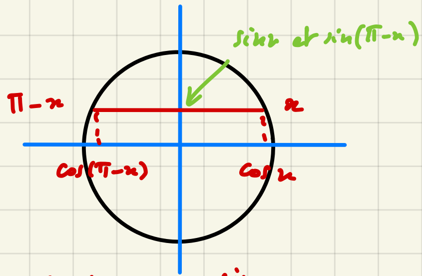
$\cos(-x) = \cos x$

$\sin(-x) = -\sin x$

$x \leftrightarrow \pi - x$ $\qquad d(\pi - x) = -dx$

$$\frac{\cos(\pi-x)}{1+\sin(\pi-x)}\,d(\pi-x) = \frac{-\cos x}{1+\sin x}(-dx)$$

$$= \frac{\cos x}{1+\sin x}\,dx \quad ✓$$



$\sin(\pi - x) = \sin x$

$\cos(\pi - x) = -\cos x$

$\Rightarrow$ on pose $t = \sin x$ $\qquad \sin 0 = 0 \qquad \sin\frac{\pi}{2} = 1$

<u>changement de variable</u>

① borner

x	0	$\frac{\pi}{2}$
t	0	1

② $dt = (\sin x)'\,dx \Rightarrow dt = \cos x\,dx$

③ on remplace tout !

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x}\,dx = \int_0^1 \frac{dt}{1+t} = \Big[\ln|1+t|\Big]_0^1 = \ln 2 - \ln 1 = \ln 2$$

Ex 2 :

$$F(x) = \int \frac{1}{\sin x}\, dx$$

$x \leftrightarrow -x$ $\qquad \frac{1}{\sin(-x)}\, d(-x) = \frac{1}{-\sin x} \times - dx = \frac{dx}{\sin x}$ ✓

$\Rightarrow$ on pose $t = \cos x$

changement de variable

① pas de bornes : c'est une primitive ! Il faudra donc revenir en x à la fin.

② $dt = (\cos x)'\, dx = -\sin x\, dx$

③ on remplace tout !

$$F(x) = \int \frac{1}{\sin x} \times \frac{\sin x}{\sin x}\, dx = \int \frac{\sin x}{\sin^2 x}\, dx = \int \frac{\sin x}{1 - \cos^2 x}\, dx$$

$\cos^2 x + \sin^2 x = 1 \Rightarrow \sin^2 x = 1 - \cos^2 x$

$$F(x) = \int \frac{-dt}{1-t^2} = \int \frac{dt}{t^2-1}$$

$$\frac{1}{t^2-1} = \frac{1}{(t-1)(t+1)} = \frac{A}{t-1} + \frac{B}{t+1} = \frac{1/2}{t-1} - \frac{1/2}{t+1}$$

$$A = \left[\frac{1}{t+1}\right]_1 = \frac{1}{2} \qquad B = \left[\frac{1}{t-1}\right]_{-1} = -\frac{1}{2}$$

$\int \frac{u'}{u} = \ln|u| + C$

$$F(x) = \int \left(\frac{1/2}{t-1} - \frac{1/2}{t+1}\right) dt = \frac{1}{2}\ln|t-1| - \frac{1}{2}\ln|t+1| + C$$

$$F(x) = \frac{1}{2}\ln|\cos x - 1| - \frac{1}{2}\ln|\cos x + 1| + C = \frac{1}{2}\ln\left(\frac{1-\cos x}{1+\cos x}\right) + C$$

$\cos x \leq 1 \Rightarrow |\cos x - 1| = 1 - \cos x$

Ex 3 :

$$I = \int \overbrace{\frac{\cos x}{2-\cos^2 x}}^{f(x)dx} dx$$

Règle de Bioche

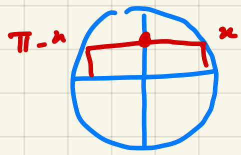
$x \longrightarrow -x$ $\qquad$ $f(x)\,dx$

$x \longrightarrow \pi - x$

$x \longrightarrow \pi + x$

$$\frac{\cos(-x)}{2-\cos^2(-x)}\,d(-x) = \frac{\cos x}{2-\cos^2 x}(-dx) \neq f(x)\,dx$$

non !



$\cos(\pi - x) = -\cos x$

$$\frac{\cos(\pi-x)}{2-\cos^2(\pi-x)}\,d(\pi-x) = \frac{-\cos x}{2-(-\cos x)^2}(-dx)$$

$$= \frac{\cos x}{2-\cos^2 x}\,dx = f(x)\,dx$$ donc on pose $t = \sin x$

changement de variable : $t = \sin x$

$$I = \int \frac{\overbrace{\cos x \, dx}^{dt}}{\underbrace{2-\cos^2 x}_{= 1+\sin^2 x \, = 1+t^2}}$$

$t = \sin x$ $\qquad$ pas de borne

$dt = \cos x \, dx$

$\cos^2 x + \sin^2 x = 1 \Rightarrow \cos^2 x = 1 - \sin^2 x$

$2 - \cos^2 x = 2 - (1 - \sin^2 x) = 1 + \sin^2 x$

$= 2 - 1 + \sin^2 x$

$$I = \int \frac{dt}{1+t^2}$$

$$I = \arctan t = \arctan(\sin x) + C$$