Exercices d'entraînements au DS2 du lundi 3/11/2019 - Corrigé

Series Entieres et Transformation de Laplace

Ex 1 :

a) $\displaystyle\sum_{n=0}^{+\infty} x^{n+1} = x \times \sum_{n=0}^{+\infty} x^n = \boxed{x \times \frac{1}{1-x}}$ si $|x|<1$ $\qquad \displaystyle\sum_{n=0}^{+\infty} X^n = \frac{1}{1-X}$ si $|X|<1$

b) $\displaystyle\sum_{n=0}^{+\infty} \frac{x^{4n}}{n!} = \sum_{n=0}^{+\infty} \frac{(x^4)^n}{n!} = \boxed{e^{x^4}}$ si $x \in \mathbb{R}$ $\qquad \displaystyle\sum_{n=0}^{+\infty} \frac{X^n}{n!} = e^X \quad \forall X \in \mathbb{R}$

c) $\displaystyle\sum_{n=1}^{+\infty} \frac{x^n}{n} = \sum_{n'=0}^{+\infty} \frac{x^{n'+1}}{n'+1}$ en posant $n' = n-1 \Leftrightarrow n = n'+1$

$$= \int \Big(\sum_{n=0}^{+\infty} x^n\Big)\,dx = \int \frac{1}{1-x}\,dx \qquad \forall |x|<1$$

$$= \boxed{-\ln|1-x| + C} \quad \text{avec } \underline{C=0} \text{ (prendre } x=0)$$

2e méthode : on dérive $\displaystyle\Big(\sum_{n=1}^{+\infty} \frac{x^n}{n}\Big)' = \sum_{n=1}^{+\infty} x^{n-1} = \sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}$ d'où $\displaystyle\sum_{n=1}^{+\infty} \frac{x^n}{n} = -\ln|1-x|$ si $|x|<1$

d) $\displaystyle\Big(\sum_{n=2}^{+\infty} \frac{x^n}{n(n-1)}\Big)' = \sum_{n=2}^{+\infty} \frac{x^{n-1}}{n-1} = \sum_{n'=1}^{+\infty} \frac{x^{n'}}{n'}$ en posant $n' = n-1$

donc $\displaystyle\sum_{n=2}^{+\infty} \frac{x^n}{n(n-1)} = \int -\ln(1-x)\,dx$ si $|x|<1$

IPP $\quad u'(x) = 1 \qquad u(x) = x$

$\quad v(x) = -\ln(1-x) \qquad v'(x) = \dfrac{1}{1-x}$

donc $\displaystyle\sum_{n=2}^{+\infty} \frac{x^n}{n(n-1)} = -x\ln(1-x) - \int \frac{x}{1-x}\,dx$ $\quad$ or $\dfrac{-x}{1-x} = \dfrac{1-x-1}{1-x} = 1 - \dfrac{1}{1-x}$

$$= -x\ln(1-x) + \int \Big(1 - \frac{1}{1-x}\Big)dx \qquad \underline{\forall |x|<1}$$

$$= \boxed{-x\ln(1-x) + x + \ln(1-x) + C} \quad \text{avec } \underline{C=0} \text{ (prendre } x=0)$$

Une série entière conserve son rayon de convergence par dérivation et intégration.

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Ex 2 :

1) $f(x) = e^x - x e^{2x}$ or $e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$ $\forall x \in \mathbb{R}$

donc $f(x) = \sum_{n=0}^{+\infty} \frac{x^n}{n!} - x \sum_{n=0}^{+\infty} \frac{(2x)^n}{n!}$ $\forall x \in \mathbb{R}$

$$f(x) = \sum_{n=0}^{+\infty} \frac{x^n}{n!} - \sum_{n=0}^{+\infty} \frac{2^n x^{n+1}}{n!}$$ car $\begin{cases} x \times x^n = x^{n+1} \\ (2x)^n = 2^n \times x^n \end{cases}$

↓

changement d'indice $n' = n+1 \Leftrightarrow n = n'-1$

$$f(x) = \sum_{n=0}^{+\infty} \frac{x^n}{n!} - \sum_{n'=1}^{+\infty} \frac{2^{n'-1} x^{n'}}{(n'-1)!}$$

$$f(x) = \underset{\substack{\downarrow \\ \text{terme d'indice } n=0}}{1} + \sum_{n=1}^{+\infty} \frac{x^n}{n!} - \sum_{n=1}^{+\infty} \frac{2^{n-1}}{(n-1)!} x^n = 1 + \sum_{n=1}^{+\infty} \left(\frac{1}{n!} - \frac{2^{n-1}}{(n-1)!} \right) x^n$$

$$f(x) = 1 + \sum_{n=1}^{+\infty} \frac{1 - n 2^{n-1}}{n!} x^n = \boxed{\sum_{n=0}^{+\infty} \frac{1 - n 2^{n-1}}{n!} x^n} \quad \forall x \in \mathbb{R}$$

2) $f(x) = \frac{1}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4}$ avec $A = \left[\frac{1}{x-4}\right]_3 = -1$

donc $f(x) = \frac{-1}{x-3} + \frac{1}{x-4}$ $\qquad B = \left[\frac{1}{x-3}\right]_4 = 1$

$$f(x) = \frac{1}{3-x} - \frac{1}{4-x} = \frac{1}{3} \frac{1}{1 - x/3} - \frac{1}{4} \frac{1}{1 - x/4}$$

or $\frac{1}{1-X} = \sum_{n=0}^{+\infty} X^n$ si $|X| < 1$

donc $f(x) = \frac{1}{3} \sum_{n=0}^{+\infty} \left(\frac{x}{3}\right)^n - \frac{1}{4} \sum_{n=0}^{+\infty} \left(\frac{x}{4}\right)^n$ si $\begin{cases} \left|\frac{x}{3}\right| < 1 \Leftrightarrow |x| < 3 \\ \text{ET} \\ \left|\frac{x}{4}\right| < 1 \Leftrightarrow |x| < 4 \end{cases}$

$$\boxed{f(x) = \sum_{n=0}^{+\infty} \left(\frac{1}{3^{n+1}} - \frac{1}{4^{n+1}} \right) x^n \text{ si } |x| < 3}$$ car $\frac{1}{3} \times \frac{1}{3^n} = \frac{1}{3^{n+1}}$

Ex 3 : (E) $x^2y'' + 4xy' + 2y = e^x$

on pose $y = \sum_{n=0}^{+\infty} a_n x^n$

$\Rightarrow y' = \sum_{n=0}^{+\infty} n a_n x^{n-1}$ et $y'' = \sum_{n=0}^{+\infty} n(n-1) a_n x^{n-2}$

e^x

$$(E) \Rightarrow x^2 \sum_{n=0}^{+\infty} n(n-1)a_n x^{n-2} + 4x \sum_{n=0}^{+\infty} n a_n x^{n-1} + 2\sum_{n=0}^{+\infty} a_n x^n = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

$$\text{soit } \sum_{n=0}^{+\infty} n(n-1)a_n x^n + \sum_{n=0}^{+\infty} 4n a_n x^n + \sum_{n=0}^{+\infty} 2a_n x^n = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

$$\Leftrightarrow \sum_{n=0}^{+\infty} \left(n(n-1)a_n + 4n a_n + 2a_n\right) x^n = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

$$\Leftrightarrow \sum_{n=0}^{+\infty} a_n \left(n^2 + 3n + 2\right) x^n = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

Deux séries entières sont égales si et ssi elles ont les mêmes coefficients !

$$\Leftrightarrow \forall n \in \mathbb{N} \quad a_n(n^2+3n+2) = \frac{1}{n!}$$

$$\Leftrightarrow \forall n \in \mathbb{N} \quad a_n(n+1)(n+2) = \frac{1}{n!} \quad \text{soit } a_n = \frac{1}{n!(n+1)(n+2)} = \frac{1}{(n+2)!}$$

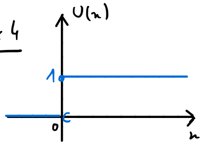
$$\text{donc } y = \sum_{n=0}^{+\infty} \frac{1}{(n+2)!} x^n = \underset{\text{si } x \neq 0}{\frac{1}{x^2}} \sum_{n=0}^{+\infty} \frac{x^{n+2}}{(n+2)!} = \frac{1}{x^2} \sum_{n'=2}^{+\infty} \frac{x^{n'}}{n'!} = \frac{1}{x^2}\left(e^x - \underset{n=0}{1} - \underset{n=1}{x}\right)$$

$n' = n+2$

Finalement, $\boxed{y = \dfrac{e^x - 1 - x}{x^2}}$ $\forall x \in \mathbb{R}^*$ et $y(0) = \frac{1}{2}$

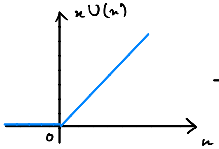
Ex 4

1)



$\xrightarrow{\mathcal{L}}$ $F(p) = \int_0^{+\infty} e^{-px}\,dx = \left[\frac{e^{-px}}{-p}\right]_0^{+\infty} = 0 + \frac{1}{p} = \boxed{\frac{1}{p}}$ si $p > 0$

car $\lim\limits_{x \to +\infty} e^{-px} = 0$



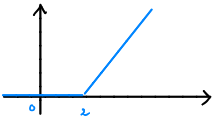
$\xrightarrow{\mathcal{L}}$ $F(p) = \int_0^{+\infty} \underbrace{x}_{v}\,\underbrace{e^{-px}}_{u'}\,dx$ IPP $\begin{cases} u'(x) = e^{-px} \\ v(x) = x \end{cases}$ $\begin{cases} u(x) = -\frac{e^{-px}}{p} \\ v'(x) = 1 \end{cases}$

$$F(p) = \underbrace{\left[-x\frac{e^{-px}}{p}\right]_0^{+\infty}}_{=0 \text{ si } p>0} - \int_0^{+\infty} -\frac{e^{-px}}{p}\,dx = \frac{1}{p}\underbrace{\int_0^{+\infty} e^{-px}\,dx}_{=\frac{1}{p}} = \boxed{\frac{1}{p^2}}$$

par croissance comparée $e^x >> x$

(cf ci-dessus)

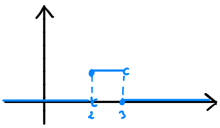
2) $f(x) = (x-2)U(x-2)$



$\xrightarrow{\mathcal{L}}$ $F(p) = \boxed{e^{-2p} \times \frac{1}{p^2}}$

$f(x-a)\,U(x-a) \xrightarrow{\mathcal{L}} e^{-ap}F(p)$

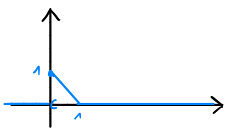
$g(x) = U(x-2) - U(x-3)$



$\xrightarrow{\mathcal{L}}$ $G(p) = \boxed{\frac{e^{-2p}}{p} - \frac{e^{-3p}}{p}}$

$h(x) = (-x+1)\big(U(x) - U(x-1)\big)$

$h(x) = (-x+1)U(x) + (x-1)\,U(x-1)$



$\xrightarrow{\mathcal{L}}$ $-\frac{1}{p^2} + \frac{1}{p} + \frac{e^{-p}}{p^2}$

Ex 5 :

$$(3x^2 - 5x + 1)\underbrace{U(x)}_{=1} \xrightarrow{\mathcal{L}} \boxed{\frac{6}{p^3} - \frac{5}{p^2} + \frac{1}{p}} \qquad x^n \xrightarrow{\mathcal{L}} \frac{n!}{p^{n+1}}$$

$$\frac{3}{p-4} \xrightarrow{\mathcal{L}^{-1}} \boxed{3e^{4x}U(x)} \quad \text{car } e^{\omega x} \xrightarrow{\mathcal{L}} \frac{1}{p-\omega} \text{ si } p > \omega$$

$$x^3 e^{2x} U(x) \xrightarrow{\mathcal{L}} \boxed{\frac{6}{(p-2)^4}} \text{ si } p-2>0 \iff p>2 \quad \text{car } \begin{cases} e^{\omega x} f(x) \xrightarrow{\mathcal{L}} F(p-\omega) \\ x^3 \xrightarrow{\mathcal{L}} \frac{6}{p^4} \text{ si } p>0 \end{cases}$$

$$(\cos(3x) - 2\sin(3x))U(x) \xrightarrow{\mathcal{L}} \frac{p}{p^2+9} - \frac{2\times 3}{p^2+9} \quad \text{car } \cos ax \xrightarrow{\mathcal{L}} \frac{p}{p^2+a^2} \text{ si } p>0$$

$$= \boxed{\frac{p-6}{p^2+9}} \text{ si } p>0 \qquad \sin ax \xrightarrow{\mathcal{L}} \frac{a}{p^2+a^2} \text{ si } p>0$$

$$\frac{p+1}{p^2-5p+6} = \frac{p+1}{(p-2)(p-3)} = \frac{A}{p-2} + \frac{B}{p-3} \qquad A = \left[\frac{p+1}{p-3}\right]_2 = -3$$

$$B = \left[\frac{p+1}{p-2}\right]_3 = 4$$

$$\text{donc } \frac{p+1}{p^2-5p+6} = \frac{-3}{p-2} + \frac{4}{p-3} \xrightarrow{\mathcal{L}^{-1}} \boxed{(-3e^{2x} + 4e^{3x})U(x)}$$

Ex 6 :

$$\begin{cases} y'' - 4y' - 5y = 0 \\ y(0) = -1 \ ; \ y'(0) = 0 \end{cases}$$

$$y(x) \xrightarrow{\mathcal{L}} Y(p)$$

$$y'(x) \xrightarrow{\mathcal{L}} pY(p) - y(0) = pY(p) + 1$$

$$y''(x) \xrightarrow{\mathcal{L}} p^2Y(p) - py(0) - y'(0) = p^2Y(p) + p$$

$$\text{donc } p^2Y(p) + p - 4(pY(p)+1) - 5Y(p) = 0$$

$$\iff (p^2 - 4p - 5)Y(p) + p - 4 = 0$$

$$\iff Y(p) = \frac{-p+4}{p^2-4p-5} = \frac{-p+4}{(p-5)(p+1)} = \frac{-1/6}{p-5} + \frac{-5/6}{p+1} \xrightarrow{\mathcal{L}^{-1}} \boxed{y(x) = \left(-\frac{1}{6}e^{5x} - \frac{5}{6}e^{-x}\right)U(x)}$$

Méthode classique : (e) $r^2 - 4r - 5 = 0 \iff (r-5)(r+1) = 0 \Rightarrow y = Ae^{5x} + Be^{-x}$

$$\begin{cases} y(0) = A + B = -1 & (L_1) \\ y'(0) = 5A - B = 0 & (L_2) \end{cases} \quad (L_1)+(L_2) \quad 6A = -1 \Rightarrow A = -\tfrac{1}{6}$$

$$\text{d'où } B = -1 - A = -1 + \tfrac{1}{6} = -\tfrac{5}{6} \qquad \boxed{y = -\tfrac{1}{6}e^{5x} - \tfrac{5}{6}e^{-x}}$$