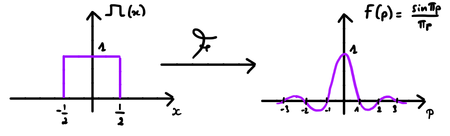
TRANSFORMATION DE FOURIER

Définition :

Si f est C^1 par morceaux et $\int_{-\infty}^{+\infty} |f(x)|\,dx$ est finie

$$\boxed{f(x) \xrightarrow{\mathcal{F}} \int_{-\infty}^{+\infty} f(x)\, e^{-2j\pi p x}\, dx}$$

On note $F = \mathcal{F}(f)$ la transformée de Fourier de f



Propriétés :

$$a f(x) + b g(x) \xrightarrow{\mathcal{F}} a F(p) + b G(p)$$

$$f(ax) \xrightarrow{\mathcal{F}} \frac{1}{|a|} F\left(\frac{p}{a}\right)$$

$$f(x - x_0) \xrightarrow{\mathcal{F}} e^{-2j\pi p x_0} F(p)$$

$$f'(x) \xrightarrow{\mathcal{F}} 2j\pi p\, F(p)$$

Formule de Parseval :

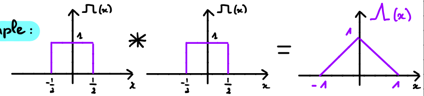
$$\int_{-\infty}^{+\infty} |f(x)|^2\, dx = \int_{-\infty}^{+\infty} |F(p)|^2\, dp$$

Produit de convolution :

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(t)\, g(x-t)\, dt$$

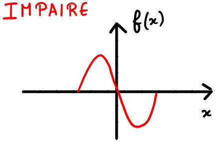
$$(f * g)(x) \xrightarrow{\mathcal{F}} F(p)\, G(p)$$

exemple :



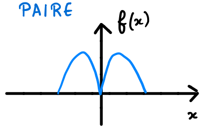
En effet, $(\Pi * \Pi)(x) \xrightarrow{\mathcal{F}} \left(\frac{\sin \pi p}{\pi p}\right)^2$ et $\Lambda(x) \xrightarrow{\mathcal{F}} \left(\frac{\sin \pi p}{\pi p}\right)^2$

IMPAIRE



$$F(p) = -2j \int_0^{+\infty} f(x) \sin(2\pi p x)\, dx$$

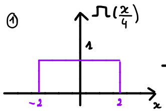
PAIRE



$$F(p) = 2 \int_0^{+\infty} f(x) \cos(2\pi p x)\, dx$$

Exemples d'Application :

①



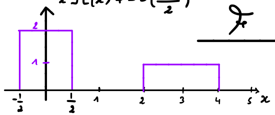
1ère méthode (calcul d'intégrale, f étant paire)

$$F(p) = 2\int_0^2 1 \cos(2\pi p x)\, dx = 2\left[\frac{\sin(2\pi p x)}{2\pi p}\right]_0^2 = \frac{\sin(4\pi p)}{\pi p}$$

2ème méthode (en utilisant la transformée de $f(ax)$)

$$F(p) = \cancel{4}\, \frac{\sin(4\pi p)}{\cancel{4}\pi p} \quad \text{car} \quad \Pi(x) \xrightarrow{\mathcal{F}} \frac{\sin(\pi p)}{\pi p}$$

② $2\Pi(x) + \Pi\left(\frac{x-3}{2}\right)$



$$F(p) = 2\,\frac{\sin(\pi p)}{\pi p} + e^{-6j\pi p} \times \cancel{2}\, \frac{\sin(2\pi p)}{\cancel{2}\pi p}$$

Transformée de Fourier inverse :

$$F(p) \xrightarrow{\mathcal{F}^{-1}} \boxed{f(x) = \int_{-\infty}^{+\infty} F(p)\, e^{2j\pi p x}\, dp}$$

L'expression est la même en échangeant :

x	p
f	F
j	$-j$

On a donc : $F(p - p_0) \xrightarrow{\mathcal{F}^{-1}} e^{2j\pi p_0 x} f(x)$

$$F'(p) \xrightarrow{\mathcal{F}^{-1}} -2j\pi x\, f(x)$$

Conséquences :

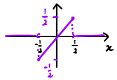
$$e^{j2\pi p_0 x} f(x) \xrightarrow{\mathcal{F}} F(p - p_0)$$

$$x\, f(x) \xrightarrow{\mathcal{F}} \frac{F'(p)}{-2j\pi}$$

De plus : Si f est paire, $\mathcal{F}(\mathcal{F}(f)) = f$

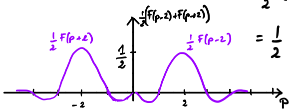
Par exemple, $\frac{\sin \pi x}{\pi x} \xrightarrow{\mathcal{F}} \Pi(p)$ car Π est paire

Exemples : ① $x\,\Pi(x) \xrightarrow{\mathcal{F}} \frac{1}{-2j\pi}\left(\frac{\sin \pi p}{\pi p}\right)' = \frac{j}{2}\left(\frac{\cos \pi p}{\pi p} - \frac{\sin \pi p}{(\pi p)^2}\right)$



② $f(x)\cos \omega x = f(x)\,\frac{e^{j\omega x} + e^{-j\omega x}}{2} \xrightarrow{\mathcal{F}} \frac{1}{2}\left(F\left(p - \frac{\omega}{2\pi}\right) + F\left(p + \frac{\omega}{2\pi}\right)\right)$

Par exemple, $\Pi(x)\cos(4\pi x) \xrightarrow{\mathcal{F}} \frac{1}{2}\left(F(p-2) + F(p+2)\right)$



$$= \frac{1}{2}\left(\frac{\sin \pi(p-2)}{\pi(p-2)} + \frac{\sin \pi(p+2)}{\pi(p+2)}\right)$$